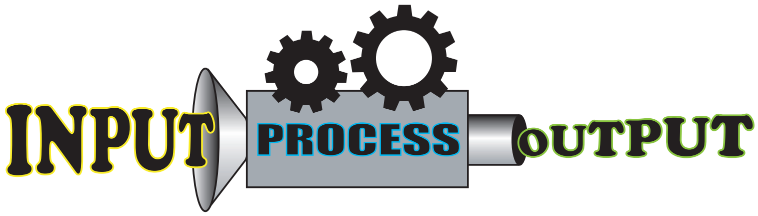
Pre-Activity Preparation

Linear Equations and Function Notation

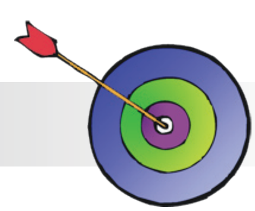


Our lives are full of input-output relationships whereby we perform some action which results in an expected outcome. Consider the following scenarios.

Input (initial action)	Process	Output (expected outcome)
Push "K" on a snack machine after inserting money	Machine mechanisms process the request	Snack "K" is dispensed
Dial 0 on a phone	Call is routed through a switchboard operator	The operator speaks to you
Throw a ball into the air	Gravity overcomes the upward velocity of the ball	The ball falls back to earth
Charge an item on a credit card	Account processing	Charge to your account
Pick up a rattlesnake	Snake reacts violently, defending itself	Snakebite
Send e-mail	Information travels through routers in cyber space	E-mail is received by recipient
Reduce food intake (go on a diet)	The body metabolizes fat	Weight loss
Turn on a television set	Set warms up and signal is received	Television picture appears
Let $x = 3$ in the equation $y = 2x + 1$	x is replaced by 3 in the equation $y = 2x + 1$	$y = 7$

You can probably think of many more similar situations. Such processes, for which there is exactly one output for every input, are called functions.

Learning Objectives



- Understand the characteristics of functions
- Represent a linear relationship in four different ways: numerically, graphically, algebraically, and in function notation
- Evaluate a function for a given value
- Identify the domain and range of linear functions

Terminology

Previously Used

equation
ordered pair
T-chart

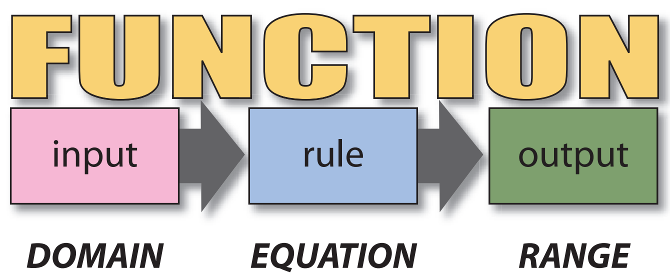
New Terms to Learn

domain
function
range

Building Mathematical Language

Functions

In mathematics, a **function** is a relationship between two sets of objects where each object in the first set pairs up with one and only one object in the second set. In the introduction, each initial action paired with one and only one outcome and each initial action had to go through some process to reach its paired outcome. A function can also have the additional requirement that each object in the first set passes through a change process to match up with its object in the second set. Look at the diagram for a visual description of function. The first set of objects is called the **domain**. The second set, the set of outcomes, is the **range**.



x	$y = -x + 3$	y	(x, y)
–2	$y = -(-2) + 3$	5	(–2, 5)
–1	$y = -(-1) + 3$	4	(–1, 4)
0	$y = -(0) + 3$	3	(0, 3)
1	$y = -(1) + 3$	2	(1, 2)
2	$y = -(2) + 3$	1	(2, 1)
3	$y = -(3) + 3$	0	(3, 0)

Think about the **T-charts** in Section 5.4. An x-value was substituted into a linear **equation** to find the y-value. The x- and y-values were written as **ordered pairs**.

The domain is the set of all possible x-values. Only six values are listed, but any real number works as well. The set of all possible y-values makes up the range. The rule, in this case the equation $y = -x + 3$, processes each input value in exactly the same way: take its opposite, then add three. Every domain value results in one and only one range value.

Determining the domain and range is a step in analyzing functions. It is important to know what the input and the output values look like. For example, if the function is $y = 20x$, you might assume that the domain is all real numbers (that is, all numbers which may be represented on a number line, and indicated by the symbol $\mathbb{R}$) because you can substitute any number for x and get a unique value as a solution. However, what if the function were to represent distance given a specific rate of 20 and x as time? The domain would then be restricted to numbers greater than 0, because time cannot be negative.

Be aware, as you work with functions, that what "works" mathematically does not always make sense in real world situations. You will see this difference demonstrated in Model 3.

Determine the domain by asking: "what are the possible replacement values for x that make sense?" Determine the range by asking "what are all the possible y-values?"

Function Notation

If an equation is a function, then for every input value there is a unique output value. Function notation takes advantage of that extra bit of information. The value of y depends of the value of x. In function notation, we replace y with $f(x)$, which is read "f of x". Let's look at how a linear function moves from standard form to slope-intercept form to function notation for x and for $x = 2$.

Standard form	Slope-intercept form	Function notation	
$x + y = 3$	$y = -x + 3$	$f(x) = -x + 3$	*general*
Let $x = 2$	Let $x = 2$	Let $x = 2$	
$(2) + y = 3$	$y = -(2) + 3$	$f(2) = -(2) + 3$	
$y = -2 + 3$	$y = 1$	$f(2) = 1$	*specific*
$y = 1$			

As values from the domain are substituted into $f(x)$, the substitution can be incorporated into the new form: $f(2) = 1$ means that 2 is substituted into the expression and 1 is the result.

In function notation, ordered pairs take on the form $(x, f(x))$. The T-chart now looks a bit different, but represents the same information.

Domain Value   Range Value

x	$f(x) = -x + 3$	$f(x)$	(x, y)
–2	$f(-2) = -(-2) + 3$	5	(–2, 5)
–1	$f(-1) = -(-1) + 3$	4	(–1, 4)
0	$f(0) = -(0) + 3$	3	(0, 3)
1	$f(1) = -(1) + 3$	2	(1, 2)
2	$f(2) = -(2) + 3$	1	(2, 1)
3	$f(3) = -(3) + 3$	0	(3, 0)

Four Ways to Represent Linear Relationships

One of the ideas mathematicians value is multiple representations of the same information. You have encountered this several times in this course and in other math courses you have taken. In some instances, like comparing numbers as fractions, decimals, and percents, or representing the same distance in metric or English units, multiple representations allowed development of your own sense of numerical order. Here we use multiple representations to take a fundamental step forward in your understanding of the function concept. We make connections among four representations of a function: numeric, graphic, algebraic, and functional notation.

Representation	
Numeric	ordered pairs listed in a table
Graphic	ordered pairs plotted and their line graphed
Algebraic	an equation in standard or slope-intercept form
Function	function notation: $f(x)$

TECHNIQUE

Translating a Linear Equation

Each of the following examples consists of ordered pairs in a linear relationship. Represent the relationship in four different ways: numerically (as ordered pairs in a table), graphically (as a graph of the set of ordered pairs), algebraically (as an equation in x and y), and in function notation. Find the domain and range of the function, and the ordered pair that is associated with the value $f(5)$.

- **Example 1**: $\{(2, 5), (-3, -5), (0, 1), (3, 7), (-2, -3), (-1, -1), (1, 3)\}$
- **Example 2**: $\{(-3, 10), (0, 3), (4, -5), (-1, 5), (1, 1), (2, -1)\}$

Try It!

<table>
<tr><th colspan="2">Type of Representation</th><th>Example 1</th><th>Example 2</th></tr>
<tr><td>Numeric</td><td>List ordered pairs in a table.

Arrange the pairs from the smallest to largest x-value.</td><td>
<table>
<tr><th>x</th><th>y</th></tr>
<tr><td>–3</td><td>–5</td></tr>
<tr><td>–2</td><td>–3</td></tr>
<tr><td>–1</td><td>–1</td></tr>
<tr><td>0</td><td>1</td></tr>
<tr><td>1</td><td>3</td></tr>
<tr><td>2</td><td>5</td></tr>
<tr><td>3</td><td>7</td></tr>
</table>
</td><td>
<table>
<tr><th>x</th><th>y</th></tr>
<tr><td></td><td></td></tr>
<tr><td></td><td></td></tr>
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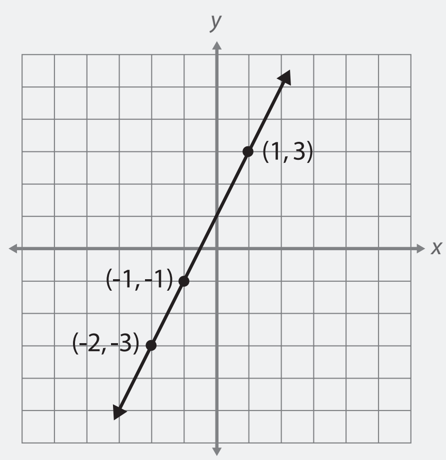
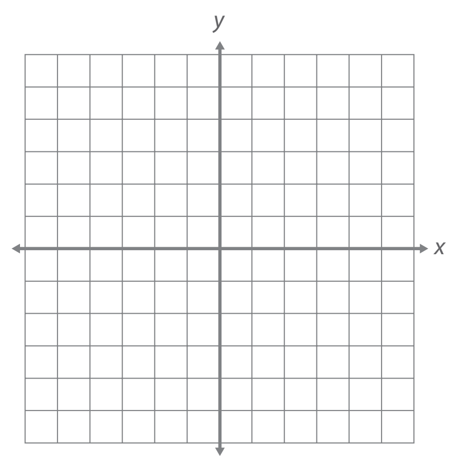
</td></tr>
<tr><td>Graphic</td><td>Plot ordered pairs on the coordinate axes.

Plot at least three points to validate the graph.</td><td>
</td><td>
</td></tr>
</table>

<table>
<tr><th colspan="2">Type of Representation</th><th>Example 1</th><th>Example 2</th></tr>
<tr><td>Algebraic</td><td>Write an equation in slope-intercept form.

Use the slope formula and the y-intercept. Write the equation.</td><td>Point 1: (1, 3)   Point 2: (–1, –1)
$m = \dfrac{y_2 - y_1}{x_2 - x_1}$
$m = \dfrac{3-(-1)}{1-(-1)} = \dfrac{4}{2} = 2$
y-intercept: (0, 1) so b = 1
equation: $y = 2x + 1$

Validate by substituting a different point (one not used in determining the slope or the y-intercept) from the table into the equation. We'll use (2, 5):
$5 \stackrel{?}{=} 2(2) + 1$
$5 = 5$ ✓</td><td></td></tr>
<tr><td>Functional</td><td>Translate the equation to function notation.

Replace y with f(x).*</td><td>$f(x) = 2x + 1$</td><td></td></tr>
<tr><th colspan="2">Problem</th><th>Example 1</th><th>Example 2</th></tr>
<tr><td colspan="2">Find the domain and range of the function.</td><td>The domain (possible x-values) is:
All real numbers ($\mathbb{R}$)

The range (possible solutions) is:
All real numbers ($\mathbb{R}$)</td><td></td></tr>
<tr><td colspan="2">Find the ordered pair that is associated with the value f(5).</td><td>$f(5) = 2(5) + 1 = 11$
The ordered pair is (5, 11).</td><td></td></tr>
</table>

* Note that when translating a linear equation to function notation, the equation must first be expressed in terms of y ($y =$).

To translate the equation $4x + 2y = 8$ to function notation, first solve for y:

$$\boxed{\begin{aligned} 4x + 2y &= 8 \\ 2y &= -4x + 8 \\ y &= -2x + 4 \end{aligned}}$$

Once we have the equation in this format, we then replace y with $f(x)$:

$$f(x) = -2x + 4$$

Models

Model 1

Evaluate the function for the given values.

Function	$f(-1)$ $f(0)$ $f(100)$	Ordered pair solution
$f(x) = 2x - 3$	$f(-1) = 2(-1) - 3 = -5$	$(-1, -5)$
	$f(0) = 2(0) - 3 = -3$	$(0, -3)$
	$f(100) = 2(100) - 3 = 197$	$(100, 197)$
$f(x) = \frac{1}{2}x + 1$	$f(-1) = \frac{1}{2}(-1) + 1 = \frac{1}{2}$	$\left(-1, \frac{1}{2}\right)$
	$f(0) = \frac{1}{2}(0) + 1 = 1$	$(0, 1)$
	$f(100) = \frac{1}{2}(100) + 1 = 51$	$(100, 51)$
Function	**g(h)** **g(☺)**	**Ordered pair solution**
$g(x) = 3x$ (While the letter f is most often used to denote a function, it is not the only letter that may be used.)	$g(\text{h}) = 3(\text{h}) = 3\text{h}$	(h, 3h)
	g(☺) = 3(☺) = 3☺	(☺, 3☺)

Note: While these last two examples do not produce ordered pairs that can be plotted in the rectangular coordinate system, they do illustrate an important feature of functions: Each input, whether a numerical value, a variable, or a symbol, leads to precisely one output. This feature of functions is one that you will encounter again as you continue your mathematical education.

Model 2

The following set of ordered pairs represent data points in a linear relationship.

$$\left\{\left(0,-\frac{3}{2}\right),(1,-1),(5,1),(3,0),(-1,-2)\right\}$$

Represent this relationship in four different ways: **numerically** (as ordered pairs in a table), **graphically** (as a graph of the set of ordered pairs), **algebraically** (as an equation in *x* and *y)*, and in **function notation**. Determine the domain and range of the data points, as well as the domain and range of the function.

numerically

Arrange the pairs from the smallest to largest x-value.

x	y
-1	-2
0	$-\frac{3}{2}$
1	-1
3	0
5	1

Domain (all x-values) of the data points:

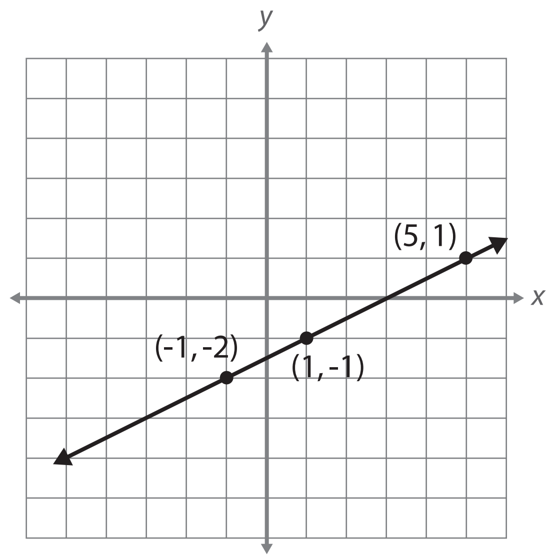
$\{-1, 0, 1, 3, 5\}$

Range (all y-values) of the data points:

$\{-2, -\frac{3}{2}, -1, 0, 1\}$

graphically

We will plot the points (–1, –2) and (1, –1) for our graph and use (5, 1) as a validation point.



algebraically

Point 1: (–1, –2) Point 2: (1, –1)

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{-1-(-2)}{1-(-1)} = \frac{1}{2}$$

y-intercept: $\left(0,-\frac{3}{2}\right)$ so b = $-\frac{3}{2}$

equation: $y = \frac{1}{2}x - \frac{3}{2}$

Validate by substituting a different point from the table into the equation. We'll use (3, 0):

$$0 \stackrel{?}{=} \frac{1}{2}(3) - \frac{3}{2}$$

$$0 = 0 \quad \checkmark$$

function notation

$$f(x) = \frac{1}{2}x - \frac{3}{2}$$

Domain (all possible values of x):

All real numbers ($\mathbb{R}$)

Range (all possible values of $f(x)$ or y):

All real numbers ($\mathbb{R}$)

Note: While it may seem that the range will always be the same as the domain for functions, consider the case of $f(x) = |x - 5|$. The domain would be all real numbers, but the range would only consist of *positive* numbers and zero.

Model 3

Represent the linear relationship described in the scenario below in four different ways: **numerically** (as ordered pairs in a table), **graphically** (as a graph of the set of ordered pairs), **algebraically** (as an equation in x and y), and in **function notation**. Determine the domain and range of the data points, as well as the domain and range of the function.

Scenario: At Paradise'n Plastic Company, revenue from tiki torches is a linear function of the number of torches sold. (The function is the number sold times the price). The following sales were reported for Memorial Day weekend:

(21, \$168), (7, \$56), (11, \$88), (18, \$144), (14, \$112).

numerically

x	y
7	56
11	88
14	112
18	144
21	168

Domain:

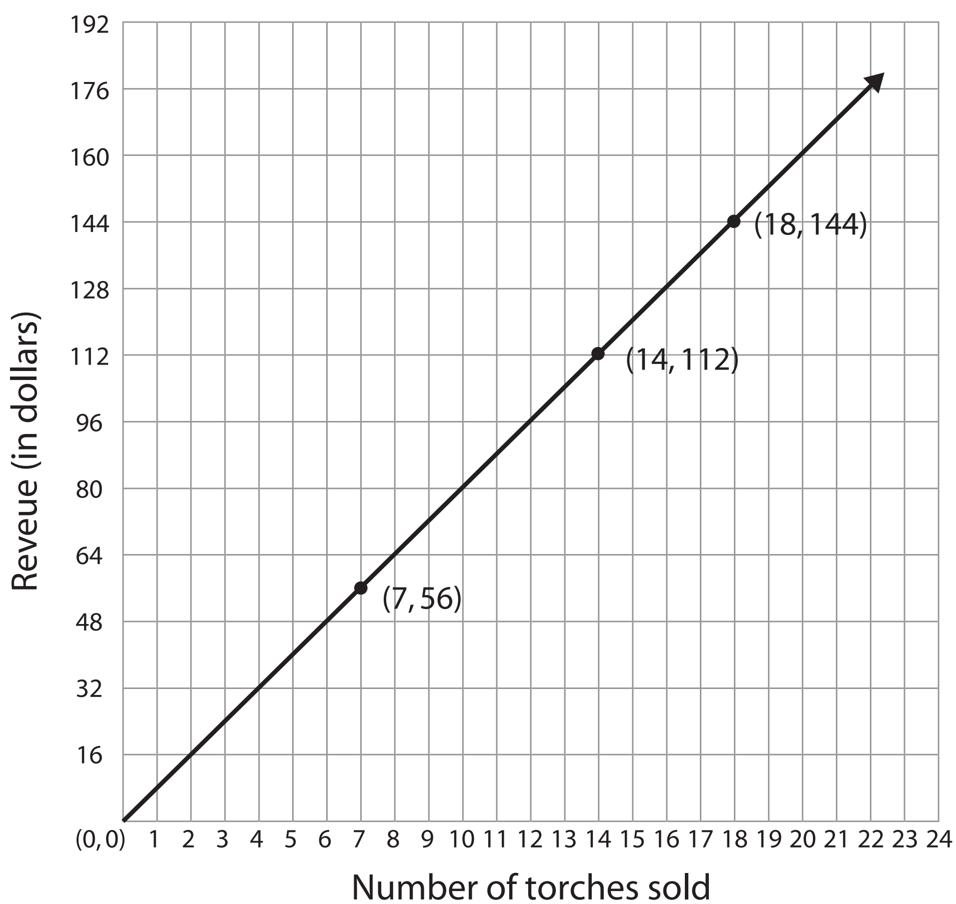
{7, 11, 14, 18, 21}

Range:

{56, 88, 112, 144, 168}

graphically

While we cannot use the normal rectangular coordinate system to plot these points and graph the relationship, we can create a graph with units that match those given in the problem. The values for the revenue (y) are large enough that it makes sense to use units larger than 1 for our graph. The y-values in the table appear to be multiples of 8, but using those units would still make for a very large graph. We use, instead, units of 16.



algebraically

We need to determine the slope of the line.

Point 1: (7, 56) Point 2: (11, 88)

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{88-56}{11-7} = \frac{32}{4} = 8$$

There is no ordered pair representing the y-intercept.

We can see from the graph that the y-intercept is 0, but we can also determine the y-intercept by reasoning, so we do not have to rely on the graph.

The relationship is between the number of torches sold (x) and the revenue from those sales (y). If no torches are sold ($x = 0$) then there will be no revenue ($y = 0$). So we know that when $x = 0$, the value of y is also 0.

This gives a y-intercept of (0, 0). So b = 0

equation: $y = 8x$

Validate by substituting a different point from the table into the equation. Use (21, 168):

$$168 \stackrel{?}{=} 8(21)$$

$$168 = 168 \quad \checkmark$$

What does a slope of 8 mean in the context of the problem?

To find out, let's translate the equation into words:

revenue = 8 × (number of torches sold)

If we sell 1 torch (let $x = 1$ in the equation $y = 8x$), the result (revenue) is \$8. This means that each torch that is sold generates revenue of \$8.

So a slope of 8 means revenue of \$8 per torch.

function notation

$$f(x) = 8x$$

Domain (all possible values of x or the number of torches that *can* be sold):

All positive numbers (there is no such thing as a negative torch)

Range (all possible values of $f(x)$ or y, which is the revenue that can be generated):

All positive multiples of 8 (there is no such thing as a negative dollar and Paradise'n Plastic does not sell half tiki torches for \$4).

Note: While this example uses real-world information, the function is actually more complex than the treatment we have given it. In point of fact, the function is not continuous—there is not a smooth progression of data points, such as the graph would lead us to believe. This is because Paradise'n Plastic does not sell fractional tiki torches. The sale of one tiki torch gives a revenue of \$8. Selling two tiki torches gives a revenue of \$16. But there is no way to generate a revenue of \$12, so the function is actually discontinuous between each number of torches. In other words, while the **equation** may be mathematically continuous (we can substitute fractions for values of x and obtain potentially any value for y), the **function** only works when x is a whole number.

Addressing Common Errors

<table>
<tr><th>Issue</th><th>Incorrect Process</th><th>Resolution</th><th>Correct Process</th></tr>
<tr>
<td>Confusing the f(x) value with the value of x</td>
<td>Name the point associated with
$f(-2) = 5$

~~(5, –2)~~</td>
<td>When translating from function notation to an ordered pair, use the form
$(x, f(x))$.</td>
<td>(–2, 5)</td>
</tr>
<tr>
<td>Confusing the domain with the range</td>
<td>What is the domain of the data set below?
{(–1, 3), (0, 2), (1, 1)}

Answer:

~~The domain is {3, 2, 1}~~</td>
<td>The domain consists of the x-values in a set of ordered pairs. The range consists of the y-values in a set of ordered pairs.</td>
<td>The domain is {–1, 0, 1}</td>
</tr>
<tr>
<td>Confusing the domain and range of data points with the domain and range of a function</td>
<td>For the data points {(–1, 3), (0, 2), (1, 1)}, give the range of the data and the range of the underlying function.

~~range: {3, 2, 1}~~</td>
<td>The range of data points (when given as ordered pairs) consists of the y-values given. The range of a function includes all possible solutions to that function, not only the ordered pairs initially given.
To find the range of a function, you must determine the function equation.</td>
<td><table><tr><th>x</th><th>y</th></tr><tr><td>–1</td><td>3</td></tr><tr><td>0</td><td>2</td></tr><tr><td>1</td><td>1</td></tr></table>The range of the data points is:
{3, 2, 1}

$\text{Slope} = \frac{3-1}{-1-1} = -\frac{2}{2} = -1$

y-intercept (0, 2)

function: $f(x) = -x + 2$

The range of the function is all real numbers ($\mathbb{R}$)</td>
</tr>
</table>

Issue	Incorrect Process	Resolution	Correct Process
Not placing a linear equation in terms of *y* before translating to function notation	Translate the following linear equation into function notation: $-4x + 2y = -2$ Answer: $\cancel{2f(x) = 4x - 2}$	Before translating a linear equation to function notation, the equation must be in terms of *y* (*y* =), where *y* has no coefficient.	Solve for y: $-4x + 2y = -2$ $2y = 4x - 2$ $y = 2x - 1$ The equation is: $y = 2x - 1$ so the function is: $\boldsymbol{f(x) = 2x - 1}$

Preparation Inventory

Before proceeding, you should be able to perform each of the following tasks:

- ❑ Represent a linear relationship numerically, graphically, algebraically, and in function notation
- ❑ Determine the domain and range of a set of data points
- ❑ Determine the domain and range of a linear function
- ❑ Determine a data point solution for a linear function, for a given input value

Section 5.5

Activity — Linear Equations and Function Notation

Performance Criteria

- Represent linear relationships numerically, graphically, algebraically, and in function notation
  - correct and appropriate notation
  - appropriate validations
- Evaluate a function for a given input value
  - accuracy
  - appropriate notation
- Determine the domain and range of a data set
  - correct reasoning
  - appropriate notation
- Determine the domain and range of a function
  - correct reasoning
  - appropriate notation

Critical Thinking Questions

1. The introduction to this section included several examples of input-output relationships that behaved like functions. Describe two additional relationships that fit the same criteria.

Input (initial action)	Process	Output (expected outcome)

2. What do the terms *domain* and *range* mean in terms of a linear equation?

3. Is the equation for a vertical line a function? Explain why or why not.

4. Is the equation for a horizontal line a function? Explain why or why not.

5. If you know a function is linear, and that its domain is all positive numbers but that its range is all negative numbers, what might the graph look like? Feel free to sketch an example, but explain your answer as well.

6. If the x-values in a table of a linear equation are increasing while the y-values are decreasing, what will the graph of the linear function look like? Feel free to sketch an example, but explain your answer as well.

Tips for Success

- Domain values are read on the x-axis of a graph
- When evaluating a function for a given input value, always write the solution as an ordered pair.

Demonstrate Your Understanding

1. Let $f(x) = -2x + 4$ and $g(x) = x + 1$. Evaluate the functions for the given values.

Value	Solution
a) $f(-1)$	
b) $g(3)$	
c) $f(P)$	
d) $f(0)$	
e) $g(-13)$	
f) $f(5)$	
g) $f(\blacktriangle)$	
h) $f(x+1)$	

2. Translate the following equations to function notation and evaluate for $f(-2)$.

Problem	Function Notation	$f(-2)$
a) $6y - 5x + 12 = 0$		
b) $x - 9 = 3y$		
c) $2(x - y) + y = y - 5$		
d) $4x = -3y + 6$		
e) $y - 5 = \frac{2}{3}(x + 6)$		
f) $x = y + 19$		

3. The following set of ordered pairs represent data points in a linear relationship.

$$\{(8, 15), (0, -1), (1, 1), (4, 7), (-3, -7)\}$$

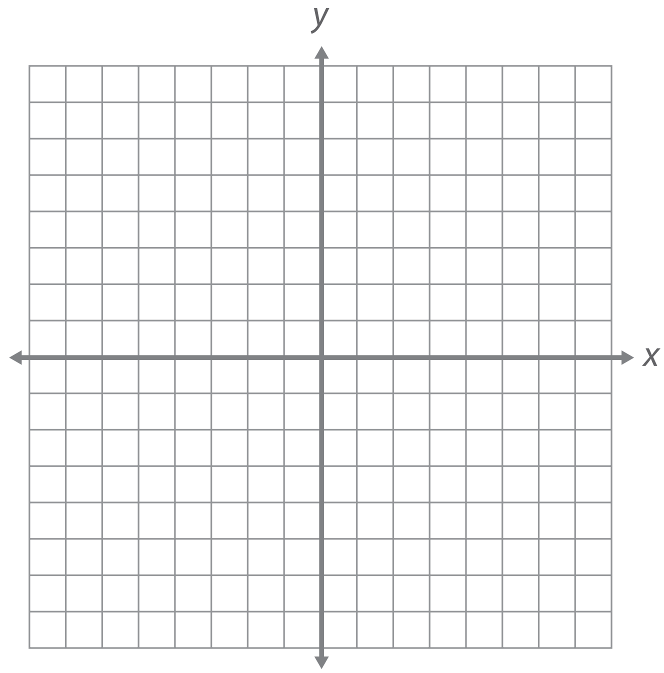
Represent this relationship in four different ways: **numerically** (as ordered pairs in a table), **graphically** (as a graph of the set of ordered pairs), **algebraically** (as an equation in x and y), and in **function notation**. Determine the domain and range of the data points, as well as the domain and range of the function.

numerically

Domain:

Range:

graphically



algebraically

function notation

Domain:

Range:

4. Represent the linear relationship described in the scenario below in four different ways: **numerically**, **graphically**, **algebraically**, and in **function notation**. Determine the domain and range of the data set, and the price per glass of lemonade.

   Scenario: Joe Schmoe is hoping to supplement his retirement income with reveue from his lemonade stand. The revenue from lemonade sales is a linear function of the number of glasses sold. (The function is the number sold times the price). Joe recorded the following sales for Monday through Friday of last week:

   {(8, \$16), (3, \$6), (5, \$10), (9, \$18), (12, \$24)}

numerically

Domain:

Range:

graphically

algebraically

Price per glass:

function notation

Identify and Correct the Errors

In the second column, identify the error(s) in the worked solution or validate its answer. If the worked solution is incorrect, solve the problem correctly.

Worked Solution	Identify Errors or Validate	Correct Process
1) What is the range of the data set below? $\{(6, 0), (-11, -2), (30, 4)\}$ Answer: **{–11, 6, 30}**		
2) Translate the following linear equation into function notation: $3x - 5y = 15$ Answer: $\mathbf{-5(f(x)) = -3x + 15}$		
3) What is the domain of the function associated with the data provided in the following T-chart? $\begin{array}{c\|c} x & y \\ \hline -1 & -1 \\ \hline 1 & 1 \\ \hline 0 & 0 \\ \hline \end{array}$ Slope $= \dfrac{1-(-1)}{1-(-1)} = \dfrac{2}{2} = 1$ y-intercept (0, 0), function: $f(x) = x$ **Domain: All real numbers**		
4) If $f(x) = 2x - 7$, what point is associated with the value $f(3)$? **(–1, 3)**		